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The spatially one-dimensional relativistic Ornstein-Uhlenbeck process in an arbitrary inertial frame

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Abstract. The spatially one-dimensional relativistic Ornstein-Uhlenbeck process is studied in an arbitrary inertial reference frame. In particular, we derive directly from the stochastic equations of motion in an arbitrary inertial frame the transport equation for the distribution function of the diffusing particles in phase-space. We explain why this result is not trivial and has, at the very least, methodological interest. We also show that this result offers a conceptually new proof of the well-known fact that the relativistic one-particle distribution function in phase-space is a Lorentz scalar.

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1 Introduction

Modeling irreversible relativistic behavior is arguably one of the most challenging issues faced today by continuous media theories and statistical physics. Indeed, if several hydrodynamical theories of relativistic fluids have been proposed during the last decades, none of them can yet be considered as the definitive relativistic generalization of, say, the Navier-Stokes model of simple fluids dynamics. As for kinetic theory, a relativistic version of Boltzmann equation does exist and has been used in various fields; nevertheless, it seems to us that it does not rest on as firm a foundation as its Galilean counterpart, if only because a simple complete statistical theory of interacting non-quantum relativistic particles is still missing, at least if the interactions are treated within the framework of field-theory.

In this context, Debbasch *et al.* [1] have recently introduced a new stochastic process which generalizes the usual Ornstein-Uhlenbeck process to the relativistic realm, and presented it as a toy-model of relativistic irreversibility. As was already mentioned in earlier publications, the special relativistic Ornstein-Uhlenbeck process, just as its Galilean counterpart, has a preferred inertial frame; this is mandatory because the physical situation at hand does exhibit a preferred reference frame, which is simply the rest-frame of the fluid in which the diffusing particles move. That the existence of a preferred inertial frame for the process does *not* contradict Einstein's relativity has been discussed at great length in references [1,2], to which we refer the interested reader. Let us just mention here, *en passant*, the Schwarzschild solution to Einstein's equations or the Friedmann-Robertson-Walker cosmological models as typical examples of purely relativistic models which trivially exhibit preferred reference frames. Let us also stress that the whole formalism associated to the ROUP is perfectly covariant, although not *manifestly* covariant. This situation, which might appear surprising, is actually quite common in physics; as another example, one can mention the canonical quantization procedure of relativistic field-theories [3].

For sheer simplicity reasons, the ROUP has been studied until now in the special relativistic framework only, and only in its preferred inertial frame. The aim of the present article is to investigate some non-trivial aspects of this stochastic process in an arbitrary inertial frame. The relativistic stochastic equations of motion which define the ROUP are non-linear (see Sect. 3), in contradistinction to their Galilean counterparts. As was already mentioned in preceding publications, this blocks a direct resolution of these equations and the main analytical tool for studying the ROUP is therefore the transport equation for the distribution function of the diffusing particles in phase space. Two common names for that equation are Kramers' equation and the forward Kolmogorov equation; in what follows, we will principally use the former denomination.

Considering the fact that Kramers' equation has already been obtained in the preferred reference frame of the ROUP [1], there are *a priori* two ways of obtaining it in an arbitrary inertial frame. The first one is to invoke the scalar nature of the phase-space distribution

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function [4–6] in order to transcribe its evolution equation, which was established in the preferred reference frame of the ROUP, in terms of the momentum and space-time coordinates associated to another, arbitrary inertial frame. The other possibility is simply to obtain the form of Kramers' equation in an arbitrary inertial frame directly from the stochastic equations of motion in that frame; this can be done, *a priori*, by following a procedure similar in spirit to the one which was already used in reference [1] to obtain Kramers' equation in the preferred frame of the ROUP from the stochastic equations of motion in that frame.

Both methods have their advantages and inconveniences. The first one is algebraically straightforward; indeed, if one uses the fact that the distribution function is a Lorentz scalar, it is nearly trivial to obtain, from the results presented in reference [1], Kramers' equation in an arbitrary inertial frame; the second method is comparatively and absolutely speaking more complicated. On the other hand, it provides an answer to a methodological question on which no light can be shed by the first method only. Indeed, the stochastic force which acts on the diffusing particle is, in the preferred reference frame of the ROUP, a Gaussian white noise (derivative of the Wiener-process) and this fact is crucial to obtaining Kramers' equation in the preferred reference frame of the ROUP [1]; however, as explained in Section 4.1, the stochastic force is no longer a Gaussian white noise in another inertial frame; it is not even a function of the time coordinate only but a function of the position occupied by the diffusing particle too. Very little is known about such position-dependent noises in general (for a good introduction to at least some aspects of the subject, the reader might wish to consult Ref. [7]); from the point of view of stochastic analysis, it is in particular absolutely *not* trivial that a simple transport equation can indeed be obtained from the stochastic equations of motion. One naturally expects however that this can be done in the case at hand, if only because such a transport equation can be derived by the alternate method mentioned earlier. But this other derivation does not instruct us about *how* to derive the transport equation in an arbitrary inertial frame directly from the equations of motion in that frame. The aim of this article is precisely to investigate this matter.

The article is organized as follows. In Section 2, we fix the major notations and, in Section 3, we review rapidly the fundamental equations which define the ROUP. In Section 4, we derive directly, by stochastic analysis only, Kramers' equation in an arbitrary inertial frame. The derivation is presented in the spatially one-dimensional case only because the three-dimensional one is far more intricate mathematically. It will be explored in a subsequent publication. We then check that our result is indeed compatible with what we would have obtained by following the first/other route. Indeed, in Section 5, we then prove directly from the result of Section 4 that the distribution function in phase-space is indeed a Lorentz scalar and discuss rapidly the initial conditions to be used in conjunction with Kramers' equation. (The proof is presented in infinite space only for simplicity reasons.) We conclude, in Section 6, by discussing our results and list some problems left open for further study. The two appendices contain some algebra which we judged too cumbersome to be included in the main part of the text.

2 Notation

In this article, the velocity of light will be denoted by c and the signature of the space time metric will be chosen to be $(+,-,-,-)$. The proper distance $\mathrm{d}s$ is therefore defined by $\mathrm{d}s^2 = (c\,\mathrm{d}t)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$.

The 4-position, 4-velocity and 4-momentum of the particle undergoing stochastic motion are denoted x^μ, $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}s$ and $p^\mu \equiv m u^\mu$ respectively. The deterministic and stochastic forces acting on the particle will be denoted f^μ and F^μ respectively. $\mathbf{v}$ is the 3-velocity of the particle defined by the usual relation $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$. γ and m are respectively the Lorentz factor and the mass of the particle. U^μ designates the (local) 4-velocity of the surrounding fluid with which the particle interacts.

As usual, Greek indices will run from 0 to 3 and Latin ones from 1 to 3.

Partial derivatives with respect to any variable a will be denoted ∂_a.

The modified Hankel function of order ν will be denoted $K_\nu(x)$ (see Ref. [8] for definitions and properties).

T will designate the absolute temperature, and k will stand for the Boltzmann constant.

In the various expansions, we have used the standard Landau's notations $\mathcal{O}(\epsilon)$ and $o(\epsilon)$ to designate quantities that are respectively of order ϵ and small compared to ϵ when ϵ vanishes.

3 Some fundamentals about the ROUP

Let us now review rapidly some general important points concerning the ROUP which will be used extensively in the rest of this article. A more substantial discussion of most of the issues raised in this section can be found in reference [1].

3.1 Definition of the ROUP

The usual Ornstein-Uhlenbeck process is a model of classical particle diffusion in which the interactions between the diffusing particle and the surrounding fluid are represented by a deterministic damping force superimposed to a random force. Physically speaking, the damping force can be interpreted as the mean effect of these interactions and the random force as the fluctuating contribution with consequently vanishing mean value. The ROUP is an extension to Einstein's relativity of this Galilean stochastic process. It is characterized by the following system of manifestly covariant equations:

$$\begin{cases} \dfrac{\mathrm{d}x^\mu}{\mathrm{d}s} = u^\mu \\ \dfrac{\mathrm{d}p^\mu}{\mathrm{d}s} = -m\lambda^\mu_\nu(u^\nu - U^\nu) + m\lambda^\alpha_\beta u_\alpha(u^\beta - U^\beta)u^\mu + F^\mu, \end{cases} \quad (1)$$

where s is the proper distance along the world line of the particle. The first two terms on the right-hand side of equation (1) define the deterministic part of the force acting on the particle. They involve a second rank tensor λ which generalizes the usual "friction-coefficient" and *a priori* depends on the thermodynamic state of the surrounding fluid and both velocities u^μ and U^μ.

From now on, we will restrict our study to special relativity and suppose that the fluid is isotropic, that it admits a global rest-frame $(\mathcal{R})$ and that its state is homogeneous and constant. For simplicity reasons, this article is devoted to the spatially one-dimensional case only. The three-dimensional one will be addressed in a forthcoming publication.

All these restrictions being made, the tensor λ, in $(\mathcal{R})$, takes the form:

$$\lambda^\mu_\nu = \begin{pmatrix} \chi(p) & 0 \\ 0 & \alpha(p), \end{pmatrix}$$

where $\alpha(p)$ is a yet unspecified function of the particle momentum in that frame [1]. A physically sensible choice of this function will be discussed in the next section. As in reference [1], the coefficient $\chi(p)$ will be chosen to be equal to zero (see the discussion in Sect. 3.1 of Ref. [1]).

The random part of the force, $F^\mu = (F^0, \gamma F/c^2)$, is characterized by the assumption that F is, in $(\mathcal{R})$, a centered Gaussian white noise which verifies:

$$\langle F(t_1)F(t_2)\rangle = -2D\delta(t_2 - t_1), \;\; D > 0.$$

More precisely, this "centered Gaussian white noise" is mathematically well-defined as a function of the time coordinate t, identical, up to a multiplicative constant $\sqrt{2D}$, to the derivative of the Wiener process $w(t)$. The latter is itself, characterized by the following properties:

(i) $w(0) = 0$.
(ii) For any $t > 0$ and for any $\Delta t \geq 0$, the random variable $\Delta w = w(t + \Delta t) - w(t)$ has the Gaussian density:

$$g(\Delta t, \Delta w) = \frac{1}{\sqrt{2\pi(\Delta t)}}\exp\left(-\frac{\Delta w^2}{2\Delta t}\right). \quad (2)$$

In $(\mathcal{R})$, system (1) can therefore be rewritten in the following differential form:

$$\begin{cases} \mathrm{d}x = v\mathrm{d}t \\ \mathrm{d}p = -\alpha(p)c\gamma p\mathrm{d}t + \sqrt{2D}\mathrm{d}w, \end{cases} \quad (3)$$

where $\gamma = \sqrt{1 + p^2/m^2c^2}$. This system is properly defined as the continuous limit, when Δt tends to zero, of the following finite difference equations:

$$\begin{cases} \Delta x = v\Delta t \\ \Delta p = -\alpha(p)c\gamma p\Delta t + \sqrt{2D}\Delta w. \end{cases} \quad (4)$$

For further convenience, we introduce $J_\mathrm{d} = -\dfrac{\alpha(p)c\gamma p\Delta t}{\sqrt{2D}}$ and $J_\mathrm{s} = \Delta w$ which fix respectively the deterministic and stochastic parts of the momentum jump in $(\mathcal{R})$.

To precisely define the stochastic process, one must add to (1) an indication about when the ROUP actually starts. In this work we will suppose that the ROUP starts at $t = 0$ in the preferred rest frame $(\mathcal{R})$ and is defined, in this frame, for all subsequent instants. This has a rather obvious physical interpretation, but some possibly unexpected consequences of this choice will be discussed in Section 5.2.

3.2 Kramers' equation in the global rest-frame of the fluid

Let $\Pi(t, x, p)$ be the phase space distribution function associated to the stochastic process (3) in $(\mathcal{R})$, the rest-frame of the fluid. As proved in reference [1], the evolution equation for Π, known as Kramers' equation, can be deduced from (3) and reads:

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p}{\gamma m}\Pi\right) + \frac{\partial}{\partial p}(-\alpha(p)c\gamma p\Pi) = D\frac{\partial^2 \Pi}{\partial p^2}\cdot \quad (5)$$

Let us here emphasize the following point: The facts that the stochastic force F is in $(\mathcal{R})$ a function of t only and that this function is essentially identical to the derivative of the Wiener process are crucial to the obtaining of (5). That both properties are not verified in other reference frames is elaborated upon in the next section, where we also show how the problem can be circumvented and how an evolution equation for the phase-space distribution in other frames can nevertheless be obtained. Such an equation is essential for a proper understanding of the ROUP because it practically constitutes the only tool for an analytical study of the process, since (3) is non-linear in p.

Equation (5) is naturally extremely difficult to solve directly. However, given an equilibrium distribution function, one can derive from (5) the corresponding expression for the coefficient $\alpha(p)$. We will suppose the relativistic Maxwell-Boltzmann equilibrium distribution[1] Π_eq to be a solution of Kramers' equation (5):

$$\Pi_\mathrm{eq} = \frac{1}{2mcK_1(\frac{mc^2}{kT})}\sqrt{\frac{kT}{mc^2}}\exp\left(-\frac{mc^2}{kT}\gamma\right). \quad (6)$$

[1] This distribution is also called Jüttner distribution.

According to reference [1], the corresponding coefficient $\alpha(p)$ takes then the following form:

$$\alpha(p) = \frac{D}{mkT}\frac{1}{\gamma^2} \equiv \frac{\alpha_0}{\gamma^2}, \quad (7)$$

where α_0 is a constitutive parameter of the model, independent of p. This result constitutes the special relativistic form of the fluctuation-dissipation theorem. With this choice for $\alpha(p)$, Kramers' equation (5) becomes:

$$\frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p}{\gamma m}\Pi\right) + \frac{\partial}{\partial p}\left(-\alpha_0 c\frac{p}{\gamma}\Pi\right) = D\frac{\partial^2 \Pi}{\partial p^2}, \quad (8)$$

and the deterministic momentum jump $J_{\rm d}$ is $-\dfrac{\alpha_0 mcv\Delta t}{\sqrt{2D}}\cdot$

4 The ROUP in an arbitrary inertial frame

4.1 The equations of motion in $(\mathcal{R}')$

Let us consider an inertial frame $(\mathcal{R}')$, deduced from the rest-frame $(\mathcal{R})$ by a global Lorentz transformation. This transformation can be characterized by the parameters $\beta = V/c$ or $\Gamma = 1/\sqrt{1-\beta^2}$, where V is the velocity of $(\mathcal{R}')$ with respect to $(\mathcal{R})$. The origins of both inertial frames are chosen so as to coincide at $t = 0$. To derive Kramers' equation in $(\mathcal{R}')$, we must first write an equivalent to system (3) in this frame. To this end, we pick a time interval $\Delta t'$ in $(\mathcal{R}')$ and consider the jump made by the particle during $\Delta t'$ in the phase space associated to $(\mathcal{R}')$. By analogy with system (3), the system describing this jump can be written as:

$$\begin{cases} \Delta x' = v'\Delta t' \\ \\ \Delta p' = \sqrt{2D}J'_{\rm d} + \sqrt{2D}J'_{\rm s}. \end{cases} \quad (9)$$

$\sqrt{2D}J'_{\rm d}$ and $\sqrt{2D}J'_{\rm s}$ denote respectively the deterministic and stochastic parts of the momentum jump in $(\mathcal{R}')$.

$\sqrt{2D}J'_{\rm d}$ is merely $f'\Delta t'$, where f' is the deterministic contribution to the force experienced by the particle in $(\mathcal{R}')$. One simple way to determine f' is to directly Lorentz transform f. The Lorentz transformation, applied to any 4-force $(\gamma\mathcal{F}v/c^3, \gamma\mathcal{F}/c^2)$ in $(\mathcal{R})$, leads to the following relation between $\mathcal{F}$ and $\mathcal{F}'$:

$$\gamma'\mathcal{F}'/c^2 = \Gamma\gamma(1-\beta v/c)\mathcal{F}/c^2.$$

Since $\Gamma\gamma(1-\beta v/c)$ is nothing else but γ', the relation between $\mathcal{F}$ and $\mathcal{F}'$ reduces to:

$$\mathcal{F}' = \mathcal{F}. \quad (10)$$

This remarkably simple relation is a direct consequence of the fact that the problem under consideration is spatially one-dimensional. When applied to f, this relation immediately delivers the desired expression for $J'_{\rm d}$:

$$J'_{\rm d} = -\frac{\alpha_0 mcv\Delta t'}{\sqrt{2D}}\cdot \quad (11)$$

$J'_{\rm d}$ must naturally be understood as a function of v'. This dependence can be made explicit by writing v as function of v' through the Lorentz transformation from $(\mathcal{R}')$ to $(\mathcal{R})$:

$$\begin{pmatrix}\gamma c\\ \gamma v\end{pmatrix} = \begin{pmatrix}\Gamma\gamma'(c+\beta v')\\ \Gamma\gamma'(v'+\beta c)\end{pmatrix}. \quad (12)$$

The expression of $J'_{\rm d}$ in terms of v' is therefore:

$$J'_{\rm d} = -\frac{1}{\sqrt{2D}}\alpha_0 cm\frac{v'+\beta}{1+\beta\frac{v'}{c}}\Delta t'. \quad (13)$$

This result can also be directly deduced from the manifestly covariant expression of the deterministic force f^μ given in (1).

Let us now concentrate on the determination of $\sqrt{2D}J'_{\rm s} = F'\Delta t'$. The most natural way to study the properties of F', the stochastic part of the force experienced by the particle in $(\mathcal{R}')$, is to use the same reasoning. However, since $t = \Gamma(t' + \beta\frac{x'}{c})$, equation (10) applied to F would yield an expression for F' that explicitly depends on both time and position in $(\mathcal{R}')$:

$$F'(x',t') = F\left(\Gamma(t'+\beta\frac{x'}{c})\right). \quad (14)$$

If only because it depends on both t' and x', this expression for F' cannot be used directly to obtain Kramers' equation in $(\mathcal{R}')$, at least by the technique presented in reference [1].

What is actually missing to derive a Kramers' equation in $(\mathcal{R}')$ is an expression for $J'_{\rm s}$ in terms of a random variable with known and "simple" properties. The only natural candidate is of course Δw. We will therefore express $J'_{\rm s}$ directly in terms of this variable. The end result will turn out to involve also $\Delta t'$ and p' and $\Delta p'$.

Let us start from (9) and (11) and first express $J'_{\rm s}$ in terms of $\Delta p'$:

$$J'_{\rm s} = \frac{1}{\sqrt{2D}}\Delta p' + \frac{1}{\sqrt{2D}}\alpha_0 mcv\Delta t'. \quad (15)$$

We then relate $\Delta p'$ to Δp through a Lorentz transformation:

$$\Delta p' = \Gamma(\Delta p - \beta mc\Delta\gamma), \quad (16)$$

where $\Delta\gamma$ is simply the jump of the Lorentz factor in $(\mathcal{R})$ associated to Δp:

$$\Delta\gamma = \left(1+\left(\frac{p+\Delta p}{mc}\right)^2\right)^{\frac{1}{2}} - \left(1+\left(\frac{p}{mc}\right)^2\right)^{\frac{1}{2}}. \quad (17)$$

Finally, we use the "original" stochastic system (4) in $(\mathcal{R})$ to express Δp in terms of Δt and Δw:

$$\Delta p = -\alpha_0 mcv\Delta t + \sqrt{2D}\Delta w. \quad (18)$$

Moreover, because $\Delta x' = v'\Delta t'$ (see (9)), the Lorentz transformation from $(\mathcal{R}')$ to $(\mathcal{R})$ relates Δt and $\Delta t'$ as follows:

$$\Delta t = \Gamma \Delta t'(1 + \beta v'/c). \tag{19}$$

Putting together equations (15) to (19), we obtain the desired expression for the stochastic momentum jump $J'_{\rm s}$:

$$J'_{\rm s} = \frac{1}{\sqrt{2D}}\Bigg\{\alpha_0 cmv\Delta t'\Big(1 - \Gamma^2(1+\beta\frac{v'}{c})\Big) + \Gamma\Big(\sqrt{2D}\Delta w - \beta mc\Delta\gamma\Big)\Bigg\}. \tag{20}$$

In the resulting expression for $J'_{\rm s}$, v and p are naturally to be understood as functions of p':

$$\begin{cases} p = \Gamma\left(p' + \beta\sqrt{m^2c^2 + p'^2}\right), \\ v = c\,\dfrac{p' + \beta\sqrt{m^2c^2+p'^2}}{\beta p' + \sqrt{m^2c^2+p'^2}}. \end{cases} \tag{21}$$

Moreover, Δp is to be understood as a function of p' and $\Delta p'$ through the inverse of the Lorentz transformation (16):

$$\Delta p = \Gamma(\Delta p' + \beta mc\Delta\gamma'), \tag{22}$$

with

$$\Delta\gamma' = \left(1 + \left(\frac{p'+\Delta p'}{mc}\right)^2\right)^{\frac{1}{2}} - \left(1 + \left(\frac{p'}{mc}\right)^2\right)^{\frac{1}{2}}. \tag{23}$$

Finally, we introduce for the random variable $J'_{\rm s}$ the natural probability density g' defined by:

$$g'(\Delta t', J'_{\rm s})\mathrm{d}J'_{\rm s} = g(\Delta t, \Delta w)\mathrm{d}\Delta w. \tag{24}$$

4.2 Derivation of Kramers' equation in $(\mathcal{R}')$

To ease further demonstrations, we rewrite system (9) into a single 2-dimensional equation. We thus introduce the condensed notation $Z' = (x', p')$ and define the three quantities:

$$\Phi'(Z') = (v', -\alpha_0 cmv),$$
$$\mathcal{J}'_{\rm s} = (0, J'_{\rm s}),$$
$$\text{and } \mathcal{D} = \mathrm{diag}(0, \sqrt{2D}).$$

The continuous limit of the stochastic system (9) when $\Delta t'$ tends to zero takes the following form:

$$Z'(t' + \Delta t') = Z'(t') + \Phi'(Z')\Delta t' + \mathcal{D}\mathcal{J}'_{\rm s}. \tag{25}$$

We introduce now the density $G'(t', \mathcal{J}'_s)$ of the stochastic process $\mathcal{J}'_{\rm s}$:

$$G'(t', \mathcal{J}'_{\rm s}) = \delta_{x''} g'(t', J'_{\rm s}), \tag{26}$$

where $\delta_{x''}$ is the Dirac distribution of the variable x''.

Let $\Pi'(t', x', p')$ be the phase space distribution function in $(\mathcal{R}')$. At this stage, we suppose that Π' is completely independent of Π. To derive the evolution equation in $(\mathcal{R}')$, we suppose that Π' is sufficiently regular for its first derivatives with respect to time and position and its second derivative with respect to momentum to exist.

Let us consider h a $\mathcal{C}^\infty$ real-valued test function with compact support included in $\mathbb{R}^2$. In order to obtain the Kramers' equation in $(\mathcal{R}')$, we identify the mean value $\langle h\rangle(t' + \Delta t')$ of the function h at $t' + \Delta t'$ and the expectation value $E\Big(h(Z'(t' + \Delta t'))\Big)$ of the random variable $h(Z'(t' + \Delta t'))$ at the same instant.

4.2.1 Evaluation of $\langle h\rangle(t' + \Delta t')$

The mean value $\langle h\rangle(t' + \Delta t')$ can be evaluated with respect to the measure defined by Π' at $t' + \Delta t'$:

$$\langle h\rangle(t' + \Delta t') = \int_{\mathbb{R}^2} h(Z')\Pi'(t' + \Delta t', Z')\mathrm{d}Z'. \tag{27}$$

Replacing $\Pi'(t' + \Delta t', Z')$ with its first order Taylor expansion, equation (27) becomes:

$$\langle h\rangle(t' + \Delta t') = \int_{\mathbb{R}^2} h(Z')\Pi'(t', Z')\mathrm{d}Z' + \Delta t' \int_{\mathbb{R}^2} h(Z')\frac{\partial \Pi'}{\partial t'}\mathrm{d}Z' + O(\Delta t'^2). \tag{28}$$

4.2.2 Evaluation of $E\Big(h(Z'(t' + \Delta t')\Big)$

We will now establish the expression of the expectation value $E\Big(h(Z'(t' + \Delta t'))\Big)$ of the random variable $h(Z'(t' + \Delta t'))$ at the instant $t' + \Delta t'$ in terms of the distribution Π' at t'. As $Z'(t')$ and $\mathcal{J}'_{\rm s}(t')$ are two independent variables, the probability density at time t' for the pair $(Z', \mathcal{J}'_{\rm s})$ is simply the product $\Pi'(t', Z')G'(\Delta t', \mathcal{J}'_{\rm s})$. We obtain:

$$E\big(h(Z'(t' + \Delta t'))\big) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h\big(Z'(t' + \Delta t')\big)G'(\Delta t', \mathcal{J}'_{\rm s})\Pi'(t', Z')\mathrm{d}Z'\mathrm{d}\mathcal{J}'_{\rm s}. \tag{29}$$

The computation of the integral over $\mathrm{d}\mathcal{J}'_{\rm s}$ in the r.h.s of (29) requires a second order Taylor expansion of $h(Z'(t' + \Delta t'))$. Using (25) and the second order Taylor expansion of h around $Z'(t')$, we obtain:

$$h(Z'(t' + \Delta t')) = h(Z'(t')) + (\Phi'(t')\Delta t' + \mathcal{D}\mathcal{J}'_{\rm s})\tfrac{\partial h}{\partial Z'} + \tfrac{1}{2}(\Phi'(t')\Delta t' + \mathcal{D}\mathcal{J}'_{\rm s})^2\tfrac{\partial^2 h}{\partial Z'^2} + R_{\rm T}, \tag{30}$$

where $R_{\rm T}$ is the remainder of the Taylor expansion:

$$R_{\rm T} = \frac{1}{3!}\epsilon^3 \left.\frac{\partial^3 h}{\partial Z'^3}\right|_{Z'+a\epsilon}, \quad 0 < a < 1, \tag{31}$$

with $\epsilon = \Phi'(t')\Delta t' + \mathcal{D}\mathcal{J}_s'$. Substituting in (29) the Taylor expansion (30) for $h(Z'(t'+\Delta t'))$ and using the definition (26), the integral over $\mathrm{d}\mathcal{J}_s'$ in equation (29) can be re-expressed as follows:

$$\int_{\mathbb{R}^2} h(Z'(t'+\Delta t'))G'(\Delta t', \mathcal{J}'_s)\mathrm{d}\mathcal{J}_s' = h(Z'(t')) + v'\frac{\partial h}{\partial x'}\bigg|_{x'}\Delta t' + \left[\frac{\partial h}{\partial p'}\bigg|_{p'} + v'\Delta t'\frac{\partial^2 h}{\partial x'\partial p'}\bigg|_{p',x'}\right] I_1 + \frac{1}{2}\frac{\partial^2 h}{\partial p'^2}\bigg|_{p'} I_2 + \int_{\mathbb{R}^2} R_{\mathrm{T}} G'(\Delta t', \mathcal{J}_s')\mathrm{d}\mathcal{J}_s', \tag{32}$$

where I_1 and I_2 are the two integrals:

$$I_1 = \int_{\mathbb{R}} (-\alpha_0 mcv\Delta t' + \sqrt{2D}J_s')g'(\Delta t', J_s')\mathrm{d}J_s', \tag{33}$$

$$I_2 = \int_{\mathbb{R}} (-\alpha_0 mcv\Delta t' + \sqrt{2D}J_s')^2 g'(\Delta t', J_s')\mathrm{d}J_s'. \tag{34}$$

The remainder R_{T} appearing in (32) will be studied separately in Appendix B.3, where its contribution will be proven to be of order $\Delta t'^{\frac{3}{2}}$.

After cumbersome algebraic manipulations described in Appendix A, the integrals I_1 and I_2 are found to amount respectively to:

$$I_1 = \Delta t'\left(-\alpha_0 cmv - \frac{2D\beta/(mc)}{2\Gamma\gamma'^3(1+\beta\frac{v'}{c})^2}\right) + O(\Delta t'^{\frac{3}{2}}), \tag{35}$$

$$I_2 = 2D\frac{\Delta t'}{\Gamma(1+\beta\frac{v'}{c})} + O(\Delta t'^{\frac{3}{2}}). \tag{36}$$

4.2.3 Kramers' equation in ($\mathcal{R}'$)

Identifying the expression (28) for the mean value of the test function h at time $t'+\Delta t'$ with the expression of the expectation value of the random variable $h(Z'(t'+\Delta t'))$ deduced from equation (32), one obtains, after substitution of (35) and (36) for I_1 and I_2:

$$\int_{\mathbb{R}^2} h(Z')\Pi'(t', Z')\mathrm{d}Z' + \Delta t'\int_{\mathbb{R}^2} h(Z')\frac{\partial \Pi'}{\partial t'}\mathrm{d}Z' = \int_{\mathbb{R}^2} h(Z')\Pi'(t', Z')\mathrm{d}Z' + \Delta t'\int_{\mathbb{R}^2}\frac{\partial h}{\partial x'}\bigg|_{x'} v'\Pi'(t', Z')\mathrm{d}Z' + \Delta t'\int_{\mathbb{R}^2}\frac{\partial h}{\partial p'}\bigg|_{p'}\left(-\alpha_0 cmv - \frac{2D\beta/(mc)}{2\Gamma\gamma'^3(1+\beta\frac{v'}{c})^2}\right)\Pi'(t', Z')\mathrm{d}Z' + \frac{\Delta t'}{2}\int_{\mathbb{R}^2}\frac{\partial^2 h}{\partial p'^2}\bigg|_{p'}\left(\frac{2D}{\Gamma(1+\beta\frac{v'}{c})}\right)\Pi'(t', Z')\mathrm{d}Z' + O(\Delta t'^{\frac{3}{2}}). \tag{37}$$

Dividing this equation by $\Delta t'$ and taking the limit of vanishing $\Delta t'$, one has then:

$$\int_{\mathbb{R}^2} h(Z')\frac{\partial \Pi'}{\partial t'}\mathrm{d}Z' = \int_{\mathbb{R}^2}\frac{\partial h}{\partial x'}\bigg|_{x'} v'\Pi'(t', Z')\mathrm{d}Z' + \int_{\mathbb{R}^2}\frac{\partial h}{\partial p'}\bigg|_{p'}\left(-\alpha_0 cmv - \frac{2D\beta/(mc)}{2\Gamma\gamma'^3(1+\beta\frac{v'}{c})^2}\right)\Pi'(t', Z')\mathrm{d}Z' + \frac{1}{2}\int_{\mathbb{R}^2}\frac{\partial^2 h}{\partial p'^2}\bigg|_{p'}\left(\frac{2D}{\Gamma(1+\beta\frac{v'}{c})}\right)\Pi'(t', Z')\mathrm{d}Z'. \tag{38}$$

Since the support of h is compact and Φ' is bounded on any compact, one deduces by integration by part that:

$$\int_{\mathbb{R}^2} h(Z')\frac{\partial \Pi'}{\partial t'}\mathrm{d}Z' = -\int_{\mathbb{R}^2} h(Z')\frac{\partial(v'\Pi')}{\partial x'}\mathrm{d}Z' - \int_{\mathbb{R}^2} h(Z') \times \frac{\partial}{\partial p'}\left[\left(-\alpha_0 cmv - \frac{2D\beta/(mc)}{2\Gamma\gamma'^3(1+\beta\frac{v'}{c})^2}\right)\Pi'(t', Z')\right]\mathrm{d}Z' + \frac{1}{2}\int_{\mathbb{R}^2} h(Z')\frac{\partial^2}{\partial p'^2}\left[\left(\frac{2D}{\Gamma(1+\beta\frac{v'}{c})}\right)\Pi'(t', Z')\right]\mathrm{d}Z. \tag{39}$$

Because equation (39) has to be true for any h, we obtain the following evolution equation for Π':

$$\frac{\partial \Pi'}{\partial t'} + \frac{\partial(v'\Pi')}{\partial x'} + \frac{\partial}{\partial p'}\left(-\alpha_0 cmv\Pi' - \frac{2D\beta/(mc)}{2\Gamma\gamma'^3(1+\beta\frac{v'}{c})^2}\Pi'\right) = \frac{1}{2}\frac{\partial^2}{\partial p'^2}\left(\frac{2D}{\Gamma(1+\beta\frac{v'}{c})}\Pi'\right). \tag{40}$$

Using the straightforward identity:

$$\frac{\partial}{\partial p'}\left(\frac{2D}{\Gamma(1+\beta\frac{v'}{c})}\right) = -\frac{2D\beta/(mc)}{\Gamma\gamma'^3(1+\beta\frac{v'}{c})^2},$$

Kramers' equation can be rewritten in the more compact form:

$$\frac{\partial \Pi'}{\partial t'} + \frac{\partial(v'\Pi')}{\partial x'} + \frac{\partial}{\partial p'}(-\alpha_0 cmv\Pi') = \frac{1}{2}\frac{\partial}{\partial p'}\left(\frac{2D}{\Gamma(1+\beta\frac{v'}{c})}\frac{\partial \Pi'}{\partial p'}\right), \tag{41}$$

where v is to be understood as function of p':

$$v = \frac{\frac{p'}{\gamma' m} + \beta c}{1 + \beta\frac{p'}{\gamma' mc}}\cdot$$

Equation (41) naturally resembles a Kramers' equation obtained from stochastic differential equations which involve a so called "multiplicative" Gaussian white noise treated by the Stratonovich calculus [9,10]. However resembling does not mean identical, and a simpler derivation of equation (41) using standard methods of the usual stochastic differential calculus does not seem possible. In addition, we have obtained equation (41) without having been confronted at any stage to any problem even remotely similar to the difficulties involved in giving a mathematically well defined meaning to a multiplicative noise. For example, there is nothing like an Ito/Stratonovich dilemma for the present situation.

5 Variance of the phase space distribution

Let (t, x) and p be a space-time position and momentum in $(\mathcal{R})$, (t', x') and p', be the corresponding space-time position and momentum in $(\mathcal{R}')$ related to (t, x) and p *via* the proper Lorentz transformation. The principle aim of the present section is to prove that $\Pi'(t', x', p')$ and the distribution function $\tilde{\Pi}(t', x', p')$ defined in $(\mathcal{R}')$ by:

$$\tilde{\Pi}(t', x', p') = \Pi(t, x, p), \tag{42}$$

verify the same evolution equation in phase space. We will also discuss which initial conditions for $\tilde{\Pi}$ and Π' have to be used to fix properly a solution of Kramers' equation in an infinite space. This section will offer proof that our stochastic analysis of the effects of a Lorentz boost on the stochastic force is indeed compatible with the expected result that the distribution function in phase-space is a Lorentz scalar.

5.1 $\tilde{\Pi}$ and Π' verify the same evolution equation

Let us rewrite the Kramers' equation (8) for the distribution function Π in $(\mathcal{R})$, with the time- and space-derivatives written in an explicitly covariant form:

$$\frac{c}{\gamma} u^\mu \partial_\mu \Pi + \frac{\partial}{\partial p}\left(-\alpha_0 c m v \Pi\right) = D\frac{\partial^2 \Pi}{\partial p^2} \cdot \tag{43}$$

In $(\mathcal{R}')$, the explicitly covariant term $u^\mu \partial_\mu \Pi$ can be easily rewritten in terms of the derivatives of $\tilde{\Pi}$ with respect to t' and x' as:

$$u^\mu \partial_\mu \Pi = \frac{\gamma'}{c}\left(\frac{\partial \tilde{\Pi}}{\partial t'} + v'\frac{\partial \tilde{\Pi}}{\partial x'}\right). \tag{44}$$

Kramers' equation (43) thus becomes in terms of $\tilde{\Pi}$:

$$\frac{\gamma'}{\gamma}\left(\frac{\partial \tilde{\Pi}}{\partial t'} + v'\frac{\partial \tilde{\Pi}}{\partial x'}\right) + \frac{\partial p'}{\partial p}\frac{\partial}{\partial p'}\left(-\alpha_0 c m v \tilde{\Pi}\right) = D\frac{\partial p'}{\partial p}\frac{\partial}{\partial p'}\left(\frac{\partial p'}{\partial p}\frac{\partial \tilde{\Pi}}{\partial p'}\right). \tag{45}$$

The factors $\frac{\gamma'}{\gamma}$ and $\frac{\partial p'}{\partial p}$ appearing in (45) both amount to $\frac{1}{\Gamma(1+\beta\frac{v'}{c})}$. The proof of this statement lies on the identity $\frac{\partial \gamma'}{\partial p'} = \frac{p'}{\gamma' m c^2}$, and on the use of the Lorentz transformations (12). Consequently the Kramers' equation (45) for $\tilde{\Pi}$ reduces to:

$$\frac{\partial \tilde{\Pi}}{\partial t'} + v'\frac{\partial \tilde{\Pi}}{\partial x'} + \frac{\partial}{\partial p'}\left(-\alpha_0 c m v \tilde{\Pi}\right) = D\frac{\partial}{\partial p'}\left(\frac{1}{\Gamma(1+\beta\frac{v'}{c})}\frac{\partial \tilde{\Pi}}{\partial p'}\right), \tag{46}$$

which is identical to the Kramers' equation (41) for Π' in $(\mathcal{R}')$.

5.2 Initial data for Π' and $\tilde{\Pi}$ in infinite space

The fact that Π' and $\tilde{\Pi}$ satisfy the same evolution equation (41)/(46) does not guarantee by itself that both distributions are identical for a given state of the system. This is quite simply due to the fact that a differential equation *per se* does not admit a unique solution and that some additional knowledge has to be added, usually in the form of initial data and boundary conditions, for the solution to be specified unambiguously. For simplicity reasons, we will now restrict the discussion by assuming that the "volume" accessible to the diffusing particles consists in the whole physical "space" $\mathbb{R}$. A solution of (8) or (41)/(46) is then completely fixed by initial conditions which specify its value for all velocities on any particular space-like hyper-surface of the two-dimensional space-time manifold [11]. The *a priori* physically natural hyper-surface suitable for fixing initial conditions for (8) has obviously $t = 0$ as equation. Let us then choose some initial conditions on this hyper-surface and follow the time evolution of the corresponding solution Π. By Cauchy's theorem, these same initial conditions can also serve as initial data for (41) and, then, fix completely a solution Π' of that equation. Obviously, $\tilde{\Pi}$ coincides for all velocities with Π' on the initial data hyper-surface and obeys the same evolution equation. It therefore follows that both distributions are identical on their definition domain. This completes the proof that the distribution function is a Lorentz scalar.

It should anyhow be stressed that the hyper-surface whose equation is $t = 0$ does not coincide with a fixed-time hyper-surface in $(\mathcal{R}')$. One can therefore wonder if a fixed time-hyper-surface in $(\mathcal{R}')$ could not also be used for specifying initial data to (41). That this cannot be done is probably most simply seen by the following argument. Let us consider the hyper-surface whose equation (in $(\mathcal{R}')$) is $t' = K$ where K is any fixed real number. In $(\mathcal{R})$, the equation of this hyper-surface is:

$$\Gamma(t - Vx/c^2) = K.$$

For non-vanishing V and for any value of K, there are therefore points on this hyper-surface whose time-coordinate in $(\mathcal{R})$ is strictly negative. It surely makes no sense to specify a value of the distribution function (for any velocity) at such points since these points, having a negative time-coordinate in $(\mathcal{R})$, actually correspond to instants which precede the start of the stochastic Markov-process (3) ; in other more physical terms, the system of diffusing particles under study has to be considered as not yet existent for points of the space-time manifold with negative time-coordinate in $(\mathcal{R})$.

6 Discussion

As a toy-model of relativistic particle diffusion, the (special) relativistic Ornstein-Uhlenbeck process has a natural preferred reference frame, the (global) rest-frame of the fluid in which the particles diffuse. Until now, the process has only been studied in that reference frame. The aim of the present article was to provide the basics for a systematic study of the process in other inertial frames. For the sake of simplicity, we have restricted our discussion to one-dimensional diffusion and worked with the simplest version of the ROUP already described in reference [1].

Because the stochastic equations of motion associated to the ROUP are, in any reference frame, non-linear in momentum, they cannot be solved directly and the only practical way of studying the process analytically is through the evolution equation for the phase-space distribution known as Kramers' equation. In the preferred reference-frame of the ROUP, this equation can be obtained in a manner absolutely similar to the one used in the Galilean case to derive the Galilean Kramers' equation from the usual Ornstein-Uhlenbeck process. This work has been presented in reference [2]. As is well-known, the derivation relies heavily on the fact that the stochastic force appearing in the Galilean equations of motion is taken to be, up to a multiplicative constant, the derivative of the Wiener-process. This property is shared by the stochastic part of the 3-force in the relativistic model, but only in the preferred reference frame of the process, where the fluid in which the particles diffuse is (globally) at rest. In other inertial frames, the relativistic transformation laws for space-time position and momentum confer a much more involved structure to the stochastic force; for example, this force always depends on both spatial position and time in other frames. This situation, which arises only in Einsteinian relativity and does not appear in the Galilean limit, renders the problem of obtaining directly Kramers' equation from the stochastic equations of motion in these frames highly non-trivial.

The principal idea behind the method developed in this article could be summed up in the following way: since the statistical properties of the stochastic part of the force in a generic inertial frame are not simple nor directly usable in deriving a Kramers' equation, the best solution is then to express directly the stochastic part of the jump in momentum space experienced by the diffusing particle during some time interval in that frame as a (possibly complicated) function of the variation of the Wiener process during the time interval which corresponds, in the preferred reference frame of the ROUP, to the original time-interval in the other frame and which can be obtained from the latter through a simple Lorentz-transformation. The statistical properties of the stochastic part of the force in a generic inertial frame can then be deduced from those of the Wiener-process in a conceptually straightforward but mathematically somewhat convoluted manner. This in turn permits the obtaining of Kramers' equation in any inertial frame. Our result has been proved compatible with what might be considered the corner-stone of usual relativistic kinetic theory, namely that the one-particle distribution function in phase-space is a Lorentz-scalar. Actually, our result can also be considered, in the context of stochastic analysis, as a direct and new proof that the distribution function associated to the ROUP is a Lorentz-scalar; let us finally note that this proof appears to be conceptually quite independent and different from other ones which have been proposed so far in the general context of relativistic kinetic theory [4,6,12].

Let us now review rapidly some natural extensions of the work presented in this paper. The most obvious one is certainly to adapt the method developed here to also obtain Kramers' equation in any inertial frame for spatially three-dimensional problems. The next step would then be to extend our results to general relativistic problems as well. Naturally, Kramers' equation permits a systematic study of the stochastic process in the so-called hydrodynamic limit. This has already been done in the (global) rest-frame of the fluid in which the particles diffuse. In particular, a relativistic diffusion equation has been obtained in this frame. The same kind of work should obviously be accomplished in other inertial frames and in general relativity as well.

Appendix A: Evaluation of I_1 and I_2

In this appendix, we will present separately the computation of the two integrals I_1 and I_2.

A.1 Evaluation of I_1

Using the definition (24) of g', I_1 becomes:

$$I_1 = \int_{\mathbb{R}} \Delta p' g(\Delta t, \Delta w) \mathrm{d}\Delta w. \qquad \text{(A.1)}$$

Substituting $\Delta p'$ by its expression (16), with Δp given by equation (18) and Δt by equation (19), and integrating over Δw, we obtain the following expression for I_1:

$$I_1 = \Gamma \left(-\alpha_0 cmv \Gamma \Delta t' \left(1 + \beta \frac{v'}{c} \right) - \beta mc L_1 \right), \qquad \text{(A.2)}$$

where L_1 is defined by:

$$L_1 = \int_{\mathbb{R}} \Delta\gamma g(\Delta t, \Delta w) \mathrm{d}\Delta w. \qquad \text{(A.3)}$$

L_1 can not be evaluated directly. In order to estimate its contribution to I_1, we resort to a second order Taylor expansion of $\Delta\gamma$:

$$\Delta\gamma = \frac{p}{\gamma m^2c^2}\Delta p + \frac{1}{2}\frac{1}{\gamma^3 m^2 c^2}\Delta p^2 + R_1, \tag{A.4}$$

where the remainder R_1 is defined by:

$$R_1 = \frac{1}{3!}\Delta p^3 \left.\frac{\partial^3\gamma}{\partial p^3}\right|_{p+a\Delta p}, \quad \text{with } 0<a<1. \tag{A.5}$$

It will be proven in Appendix B.1 that the contribution of R_1 to I_1 is of order $\Delta t'^{\frac{3}{2}}$.

With the expression (18) for Δp and (19) for Δt, L_1 becomes:

$$L_1 = \Gamma\Delta t'\left(1+\beta\frac{v'}{c}\right)\left(\frac{-\alpha_0 cmvp}{m^2c^2\gamma} + \frac{2D}{2m^2c^2\gamma^3}\right) + O(\Delta t'^{\frac{3}{2}}). \tag{A.6}$$

Substituting this expression in (A.3), one gets finally for I_1:

$$I_1 = \Delta t'\left(-\alpha_0 cmv - \frac{2D\beta}{2\Gamma mc\gamma'^3(1+\beta\frac{v'}{c})^2}\right) + O(\Delta t'^{\frac{3}{2}}). \tag{A.7}$$

A.2 Evaluation of I_2

A procedure identical to the one used in the preceding section will be now applied in order to estimate the integral I_2. Using Identity (24) and substituting $\Delta p'$ by its expression (16), I_2 becomes:

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} (\Delta p')^2 g(\Delta t, \Delta w)\mathrm{d}\Delta w \\ &= \Gamma^2\Bigg(2D\Gamma\Delta t'\left(1+\beta\frac{v'}{c}\right) + (\beta mc)^2 L_2 \\ &\quad -2\sqrt{2D}\beta mc M_2\Bigg) + O(\Delta t'^{\frac{3}{2}}), \end{aligned} \tag{A.8}$$

where L_2 and M_2 are defined by:

$$L_2 = \int_{\mathbb{R}} (\Delta\gamma)^2 g(\Delta t, \Delta w)\mathrm{d}\Delta w, \tag{A.9}$$

$$M_2 = \int_{\mathbb{R}} \Delta w \Delta\gamma\, g(\Delta t, \Delta w)\mathrm{d}\Delta w. \tag{A.10}$$

The expression (17) for $\Delta\gamma$ and the computation (A.6) of L_1 lead to the following expression for L_2:

$$L_2 = \frac{2D}{(mc)^2}\Gamma\frac{(\frac{v'}{c}+\beta)^2}{1+\beta\frac{v'}{c}}\Delta t' + O(\Delta t'^{\frac{3}{2}}). \tag{A.11}$$

To obtain M_2, the same procedure as the one used to evaluate L_1 is followed and we multiply the Taylor expansion of $\Delta\gamma$ obtained in equation (A.4) by Δw and integrate the resulting equation over Δw. The expression deduced for M_2 is then:

$$M_2 = \sqrt{2D}\Gamma\frac{\frac{v'}{c}+\beta}{mc}\Delta t' + O(\Delta t'^{\frac{3}{2}}). \tag{A.12}$$

Note that the Taylor expansion generates a new remainder $R_2 \equiv \Delta w R_1$, which is of order $\Delta t'^{\frac{3}{2}}$ (see Appendix B.2).

Substituting L_2 by its expression (A.11) and M_2 by equation (A.12), I_2 becomes:

$$I_2 = 2D\frac{\Delta t'}{\Gamma(1+\beta\frac{v'}{c})} + O(\Delta t'^{\frac{3}{2}}). \tag{A.13}$$

Appendix B: Evaluation of the remainders

We will justify in this appendix that the remainders R_{T}, R_1 and R_2 do not give any constant or linear contribution in $\Delta t'$ to the Kramers' equation in $(\mathcal{R}')$.

In order to facilitate the reading of the following algebraic developments, we state some useful relations verified by the density g:

$$\begin{aligned} &\int_{\mathbb{R}} \Delta w^{2n} g(\Delta t, \Delta w)\mathrm{d}\Delta w = \frac{(2n)!}{2^n n!}(\Delta t)^n, \quad n\in\mathbb{N}, \\ &\int_{\mathbb{R}} \Delta w^{2n+1} g(\Delta t, \Delta w)\mathrm{d}\Delta w = 0, \quad n\in\mathbb{N}. \end{aligned} \tag{B.1}$$

B.1 The remainder R_1

R_1 is defined by

$$R_1 = \frac{1}{3!}(\Delta p)^3 \left.\frac{\partial^3\gamma}{\partial p^3}\right|_{p+a\Delta p}, \quad 0<a<1, \tag{B.2}$$

where

$$\left.\frac{\partial^3\gamma}{\partial p^3}\right|_{p+a\Delta p} = \frac{-3}{m^4c^4}\frac{p+a\Delta p}{(1+(\frac{p+\Delta p}{mc})^{\frac{5}{2}}}.$$

Using Identity (24), one obtains:

$$\begin{aligned} &\left|\int_{\mathbb{R}} R_1 g'(\Delta t', J'_{\mathrm{s}})\mathrm{d}J'_{\mathrm{s}}\right| \le \\ &\frac{1}{2m^4c^4}\int_{\mathbb{R}} |\Delta p|^3\left(|p|+|\Delta p|\right) g(\Delta t, \Delta w)\mathrm{d}\Delta w. \end{aligned} \tag{B.3}$$

The expression (18) for Δp and (19) lead to:

$$\begin{aligned} |\Delta p|^3 &\le 3\alpha_0 c|v|\Delta t'(\sqrt{2D}\Delta w)^2\Gamma\left(1+\beta\tfrac{v'}{c}\right) \\ &\quad + |\sqrt{2D}\Delta w|^3 + O(\Delta t'^{\frac{3}{2}}), \\ |\Delta p|^4 &\le 4\alpha_0 c|v|\Delta t'|\sqrt{2D}\Delta w|^3\Gamma\left(1+\beta\tfrac{v'}{c}\right) \\ &\quad + (\sqrt{2D}\Delta w)^4 + O(\Delta t'^{\frac{3}{2}}). \end{aligned} \tag{B.4}$$

With equations (B.1) and the following property:

$$\frac{1}{\sqrt{2\pi\Delta t}}\int_{\rm I\!R}|\Delta w|^3\exp\left(-\frac{(\Delta w)^2}{2\Delta t}\right)\mathrm{d}\Delta w = \frac{1}{\sqrt{\pi}}\left(2\Gamma\left(1+\beta\frac{v'}{c}\right)\Delta t'\right)^{\frac{3}{2}}, \quad \text{(B.5)}$$

we estimate the contribution of R_1 to be:

$$\left|\int_{\rm I\!R} R_1 g'(\Delta t', J'_{\rm s})\mathrm{d}J'_{\rm s}\right| \le O(\Delta t'^{\frac{3}{2}}). \quad \text{(B.6)}$$

B.2 The remainder R_2

R_2 is defined by: $R_2 = \Delta w R_1$. From (B.3), we obtain:

$$\left|\int_{\rm I\!R} R_2 g(\Delta t, \Delta w)\mathrm{d}\Delta w\right| \le \frac{1}{2m^2c^2}\int_{\rm I\!R}\left(|p||\Delta p|^3|\Delta w| + |\Delta p|^4|\Delta w|\right)g(\Delta t,\Delta w)\mathrm{d}\Delta w. \quad \text{(B.7)}$$

Using equations (B.4), the properties (B.1) and (B.5), and

$$\frac{1}{\sqrt{2\pi\Delta t}}\int_{\rm I\!R}|\Delta w|^5\exp\left(-\frac{(\Delta w)^2}{2\Delta t}\right)\mathrm{d}\Delta w = \frac{1}{\sqrt{2\pi}}(2\Delta t)^{\frac{5}{2}} = O(\Delta t'^{\frac{5}{2}}), \quad \text{(B.8)}$$

Equation (B.7) becomes:

$$\left|\int_{\rm I\!R} R_2 g(\Delta t, \Delta w)\mathrm{d}\Delta w\right| \le O(\Delta t'^{\frac{3}{2}}). \quad \text{(B.9)}$$

B.3 The remainder $R_{\rm T}$

Let us now focus our attention to $R_{\rm T}$. Using its expression (31), equation (24) and Properties (B.1), one finds:

$$\int_{\rm I\!R^2} R_{\rm T} G'(\Delta t', \mathcal{J}'_{\rm s})\mathrm{d}\mathcal{J}'_{\rm s} = \frac{1}{3!}\left.\frac{\partial^3 h}{\partial p'^3}\right|_{Z'+a\epsilon}\int_{\rm I\!R}(\Delta p')^3 g(\Delta t,\Delta w)\mathrm{d}\Delta w + \frac{1}{3!}\left.\frac{\partial^3 h}{\partial x'\partial p'^2}\right|_{Z'+a\epsilon} \times(-3\alpha_0 c v\Delta t')\int_{\rm I\!R}(\Delta p')^2 g(\Delta t,\Delta w)\mathrm{d}\Delta w + O(\Delta t'^{\frac{3}{2}}), \quad \text{(B.10)}$$

where a lies in $]0,1[$. We have obtained in Section A.1, the expression (35) for I_1 which is linear in $\Delta t'$, this implies that the second term of the right-hand of (B.10) is of order $\Delta t'^2$. Consequently, the evaluation of $R_{\rm T}$ reduces to the study of the first term of the right-hand side of (B.10).

We begin with the expansion of $(\Delta p')^3$, which reads:

$$\begin{aligned}(\Delta p')^3 = \Gamma^3\Big[&(\sqrt{2D}\Delta w)^3 - (\beta mc)^3(\Delta\gamma)^3\\ &+ 3(\sqrt{2D}\Delta w)^2\left(-\alpha_0 cmv\Gamma\Delta t'\left(1+\beta\tfrac{v'}{c}\right)\right)\\ &- 3\alpha_0 cmv\Gamma\Delta t'(1+\beta\tfrac{v'}{c})(\beta mc\Delta\gamma)^2\\ &+ 3(\sqrt{2D}\Delta w)^2(-\beta mc)\Delta\gamma\\ &+ 6\sqrt{2D}\Delta w\alpha_0 cmv\Gamma\Delta t'(1+\beta\tfrac{v'}{c}(\beta mc)\Delta\gamma)\Big].\end{aligned} \quad \text{(B.11)}$$

Multiplying (B.11) by $g(\Delta t, \Delta w)$ and integrating over Δw with the properties (B.1) for g, one gets:

$$\begin{aligned}\int_{\rm I\!R}(\Delta p')^3 g(\Delta t,\Delta w)\mathrm{d}\Delta w = \\ \Gamma^3\Big[&-(\beta mc)^3 L - 3\beta mc(2D)M\\ &- 3\alpha_0 cmv\Gamma\Delta t'\left(1+\beta\frac{v'}{c}\right)(\beta mc)^2 L_2\\ &+ 6\sqrt{2D}\alpha_0 cmv\Gamma\Delta t' M_2\Big] + O(\Delta t'^{\frac{3}{2}}).\end{aligned} \quad \text{(B.12)}$$

As seen in equations (A.11) and (A.12) respectively, L_2 and M_2 are of order $\Delta t'$ the evaluation of (B.12) reduces then to the evaluation of L and M, which are defined by:

$$L = \int_{\rm I\!R}(\Delta\gamma)^3 g(\Delta t,\Delta w)\mathrm{d}\Delta w, \quad \text{(B.13)}$$

$$M = \int_{\rm I\!R}(\Delta w)^2\Delta\gamma g(\Delta t,\Delta w)\mathrm{d}\Delta w. \quad \text{(B.14)}$$

Concentrating on L, we rewrite $\Delta\gamma^3$ with expression (17) under the form:

$$\Delta\gamma^3 = \left(1+\left(\frac{p+\Delta p}{mc}\right)^2\right)^{\frac{3}{2}} + 3\gamma^2\left(1+\left(\frac{p+\Delta p}{mc}\right)^2\right)^{\frac{1}{2}} - 3\gamma\left(1+\left(\frac{p+\Delta p}{mc}\right)^2\right) - \gamma^3. \quad \text{(B.15)}$$

The term $\left(1+\left(\frac{p+\Delta p}{mc}\right)^2\right)^{\frac{3}{2}}$ has the Taylor expansion:

$$\left(1+\left(\frac{p+\Delta p}{mc}\right)^2\right)^{\frac{3}{2}} = \gamma^3 + \Delta p\left(3\frac{\gamma p}{m^2c^2}\right) + \frac{(\Delta p)^2}{2}3\frac{\gamma}{m^2c^2}\left(1+\frac{p^2}{\gamma^2m^2c^2}\right) + S, \quad \text{(B.16)}$$

where S is defined by:

$$S = \frac{(\Delta p)^3}{3!}\left.\frac{\partial^3\gamma}{\partial p^3}\right|_{p+a\Delta p}, \quad a\in]0,1[, \quad \text{(B.17)}$$

$$\text{with}\quad \left.\frac{\partial^3\gamma}{\partial p^3}\right|_p = \frac{3p}{m^4c^4\gamma^3}\left(1-\frac{2p^2}{m^2c^2}\right). \quad \text{(B.18)}$$

Following the same reasoning used to estimate the contribution of R_1 and R_2, we find that:

$$\left|\int_{\mathbb{R}} S\, g(\Delta t, \Delta w)\mathrm{d}\Delta w\right| \leq \frac{1}{2m^4c^4}\int_{\mathbb{R}} |\Delta p|^3\,(|p| + |\Delta p|) \times \left(1 + 2\frac{|p| + |\Delta p|^2}{m^2c^2}\right) g(\Delta t, \Delta w)\mathrm{d}\Delta w, \leq O(\Delta t'^{\frac{3}{2}}). \tag{B.19}$$

Considering the result (B.19) and substituting the expression (B.16) in (B.15), one obtains finally with properties (B.1), that L is of order $\Delta t'^{\frac{3}{2}}$.

We study now the contribution of M. With the Taylor expansion (A.4) for $\Delta\gamma$, M can be rewritten:

$$\begin{aligned} M = & \frac{p}{\gamma m^2c^2}\int_{\mathbb{R}} (\Delta w)^2 \Delta p g(\Delta t, \Delta w)\mathrm{d}\Delta w \\ & + \frac{1}{2\gamma^3 m^2 c^2}\int_{\mathbb{R}} (\Delta w)^2 (\Delta p)^2 g(\Delta t, \Delta w)\mathrm{d}\Delta w \\ & + \int_{\mathbb{R}} (\Delta w)^2 R_1 g(\Delta t, \Delta w)\mathrm{d}\Delta w. \end{aligned} \tag{B.20}$$

Using the properties (B.1) and the expression (B.2) for R_1, one finds that M is of order $\Delta t'^{\frac{3}{2}}$. As L and M are of order $\Delta t'^{\frac{3}{2}}$, it follows from equation (B.12) that $\int R_{\mathrm{T}} G'(\Delta t', \mathcal{J}'_{\mathrm{s}}) d\mathcal{J}'_{\mathrm{s}}$ is of order $\Delta t'^{\frac{3}{2}}$.

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